

AN
EXAMINATION
OF THE
FIRST SIX BOOKS
OF
EUCLID'S ELEMENTS.

BY

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EXAMINATIONS
JOHN WILLIAM M.D.

OF THE
FACULTY OF MEDICINE
OF THE UNIVERSITY OF CHICAGO
IN THE DEPARTMENT OF MEDICINE
AND SURGERY

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TO
JOHN SMITH, M. D.
SAVILIAN PROFESSOR OF GEOMETRY
IN THE UNIVERSITY OF OXFORD,
THIS EXAMINATION
OF THE FIRST SIX BOOKS
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EUCLID'S ELEMENTS
IS VERY RESPECTFULLY INSCRIBED,
BY
HIS MUCH OBLIGED, AND
VERY HUMBLE SERVANT
THE AUTHOR.

JOHN SMITH, M.D.

PROFESSOR OF CHEMISTRY

IN THE UNIVERSITY OF OXFORD

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OF THE FIRST SIX BOOKS

OF LUCIUS AELIUS

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P R E F A C E.

THE following examination of the first six books of Euclid's elements is only a small effort towards a plan for facilitating the study of geometry.

No method ever proposed on this subject has been so much approved of as the elements of Euclid, nor has any other science afforded a more perfect example of abstracted reasoning; this system therefore must be esteemed the best model we can follow. Yet let it not be thought presumptuous to offer improvements upon it; let it be remembered that the works of great men are often in some parts imperfect, and leave room for inferior talents to correct and improve them.

There is one passage in the elements, in which the admirers of Euclid have all acknowledged that he failed. In the theory of parallel lines he seems to have forgotten his own principles of reasoning; and many other
geo-

geometricians, falling into the same error, have treated that subject with no better success.

Besides, we must expect to find some innovations in a work, which, for the space of two thousand years, has been constantly in the hands of teachers in this science; who had the best opportunities to discover its faults, and the strongest temptations to add, to alter, or to abridge. That this has often been done, even by the Greeks, has been long suspected; and will appear very evidently from some parts of the following examination, where the innovator will be shewn not to have imitated with sufficient care the chaste language of the original author. It is not impossible, that some faults may have been corrected; certain it is, that no inconsiderable number has been committed by the successors of Euclid.

The propensity of men to attempt emendations of works which they have long considered, and of which they think themselves masters, is sufficiently proved by the various additions and interpolations which have crept into the modern commentaries and translations of Euclid. Yet upon the whole it does
not

not appear that this science is much indebted to these innovations: and as the elements now stand, it is rather to be regretted that they should always be read in translations; or that the translations do not approach nearer to the original.

But it is not so easy to detect or to avoid the corruptions made in the original language. The writings of Euclid upon other subjects, and of the Greek mathematicians, who were cotemporary with, and subsequent to him, are the only guides, on which we can rely, for some centuries after our author. The elegance of the ancient mathematicians, the peculiar delight of the Greeks, though less regarded in the present age, will render an attentive perusal of them both an entertaining and useful employment.

The demolition of the Alexandrian library in the seventh century, has not proved so unfavourable to Euclid, as to Grecian literature in general. The Arabic versions of his elements may be compared with the Grecian copies, and afford a good criterion of the alterations they have undergone subsequent to that event.

It

It is not presumed that this examination is in any degree a perfect execution of the plan here proposed. The author is conscious of his own insufficiency for such a work; and would wish to have better qualified himself by a longer perseverance in these pursuits. But engagements of a different nature oblige him to desist from an undertaking, which he thinks worthy the attention of some person of more leisure, learning, and abilities.

AN
EXAMINATION
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EUCLID'S ELEMENTS.

BOOK I.

DEFINITIONS.

DEFINIT. I. Σ ΗΜΕΙΟΝ ἔστιν, ὃ μέρος οὐδέν. "A
 "Point is that, of which there
 "is no Part;" i. e. which
 cannot be parted or divided;
 as it is interpreted by Proclus, Ἀμέρες τι σημεῖον ὃ χω-
 μετρῆς ἐν αὐτῇ δυνάμει: And likewise by Tartalea in his
 Note upon this Definiton, "Il Punto è quello, che
 "non ha parte alcuna; cioe, quello del quale non si
 "puo tuoglier, ne trovar, ne anchora imaginar la
 "la mettade, over il terzo &c." — Theon, the
 Smyranean, and after him several of our modern
 Translators have interpreted it thus; "A Point is
 "that, which has no Parts, or Magnitude:" Which
 I do not think either so rational or so intelligible as
 the original Definition.

A

DEFINIT.

DEFINIT. 4. *Εὐθεία γραμμή ἐστίν, ἣτις ἐξίστη τοῖς ἐν αὐτῇ σημείοις κείται.* "A straight Line is that, which
 "lies upon an Equality between it's Points."

The Word *equal* is nowhere explain'd in the original Elements of Euclid: The Reason of this Omission will be seen, when we come to examine the Axioms. But agreeably to the common Acceptation of this Word, Magnitudes, which coincide or fill the same space, are said to be equal, and in no other Signification is this Word employ'd throughout the Elements of Geometry. Let it therefore have the same Meaning in this Definition; and let "straight Lines be those, which coincide between their Points." — It may be asked, why was it not expressed, that "straight Lines which coincide in two Points shall coincide in every Point?" Evidently, that their Direction might not be confounded with their Quantity: for it might be inferred from such a Definition, that all straight Lines, which coincide in two Points are equal. The Author therefore prudently expresses himself, that they are upon an Equality or Coincidence between their Points, or that straight Lines are those, which, if they coincide in any two Points, shall coincide between those Points. But as this Definition has seldom been properly understood, which is a Proof, that it is not so clear as a Definition ought to be; and as it does not express in it's full Extent that property of straight Lines, which it was intended to explain, it would perhaps be better thus; "Straight Lines are those, which, if they coincide in any two Points, coincide as far as they are produced."

That Property of straight Lines contained in the 10th Axiom, namely, "That two straight Lines
 cannot

"cannot enclose the Space," is evidently implied in this Definition, for if they enclosed a Space, they could not coincide between their Points. We have Proclus's Testimony, that this Axiom is not Euclid's, nor consistent with the Nature of an Axiom. — From the above Definition of a straight Line follows another Maxim in Geometry, that two Points fix a straight Line, *των περατων μενοντων και αυτη μνησται*; which is the Foundation of the 1st and 2d Postulates. Hence straight Lines, which are proved to coincide in any Number of Points, are called "one and the same Line." Prop. 14. Book I. or, which is the same Thing, that "two straight Lines cannot have a common Segment;" which is Simson's Corollary to Prop. 11. B. I.

DEFINIT. 7. *Επιπεδον εμφανεια εστιν, ητις εξισοιται ταις εφ' αυτης ευθειαις κειται.* "A plane Surface is that, which "lies upon an Equality with the straight Lines in "it;" in other Words, A plane Surface is that, in which the straight Line joining any two Points, lies wholly in it: which is perhaps somewhat more intelligible than a literal Translation. In this Definition, as also in that of a straight Line, the Author of the Elements by Equality means Coincidence.

The Definitions of a straight Line and a plane Surface being thus understood, the first Proposition of the 11th Book, viz. That one Part of a straight Line cannot be in a Plane and another above it, will be found to be totally unnecessary: and that it was originally not in the Elements of Euclid, we have very strong presumptive Proof from Proclus; who, speaking of Euclid's Definition of a straight Line, deduces from it, as an obvious Property, "*οτι μεροσ μεν εκ εστιν αυτης εν υποκειμενω επιπεδω, μεροσ δε εν μετεωροτερω.*" A partial Commentator of Euclid cannot be supposed to differ thus without

the least Intimation of it from his Author, and confidently to assert, as a Property obviously implied in a Definition, what Euclid had thought proper to explain by a Demonstration. It is probable therefore, that this unnecessary Proposition was inserted by some later Commentator, who did not understand these Definitions of Euclid. The Demonstration given of it is a very lame one, and has justly been objected to. Conscious of it's Defects, Simson has added a Corollary to Proposition 11. Book I. purposely to complete that Demonstration. But his Corollary is contained in the Definition of a straight Line, and the Proposition itself in that of a plane Surface.

DEFINIT. 8. "A Plane Angle is the Inclination "of two Lines to one another in a plane, which "meet together, but are not in the same Direction." The general Definition of an Angle has been often objected to: The Expression in Whiston's Edition, "but are not in the same Line," is certainly more proper: though I think it would be more intelligible to Learners, if it were expressed thus, "A Plane Angle is the Inclination of two "Lines to one another in a plane, which meet in "one Point only." That the Extremity of a Line is a Point, and that the Intersection of Lines, which make Angles, is a Point, is every where supposed in the Elements, and may therefore with great propriety be assumed in this Definition.

DEFINIT. 29 — 33 inclusively are very exceptionable. It cannot be admitted as a Definition, that all the three Angles of a Triangle are acute, which is supposed in the 29th Definition. The Angles of a Triangle must be compared with right Angles, before we can assert, that they are all less than three right Angles. The Definitions of quadrilateral

drilateral Figures, which follow, are liable to the same Objections. In the Definition of a Square, it is assumed, that all it's Sides are equal, and it's Angles right Angles: it might with the same Propriety have been said of an equilateral Triangle, that it's Sides and also it's Angles are equal to each other. But a still more weighty Objection to these Definitions is, that they are altogether unnecessary. As Savile himself confesses in his 7th Lecture, “Nec dissimulandum aliquot harum in
“manibus exiguum esse usum in Geometriâ.”

DEFINIT. 35. “Parallel straight Lines are such as are in the same Plane, and which being produced ever so far both Ways do not meet.” We shall examine this Definition, when we come to the 12th Axiom.

A X I O M S.

IT has often been questioned, whether Demonstrations might not be found of the Axioms of Euclid. It was Aristotle's opinion, that there are certain Principles in Science called Axioms, which are not demonstrable. But this opinion seems to have been suggested to him by a Matter of Fact; the Axioms in Geometry having never been demonstrated in his Time. For in another Place he makes this general Observation, † “that he, who
“would attain to Knowledge by Demonstration,
“should not only have a more perfect Knowledge
“of, and fuller Confidence in, the first Principles,
“than in the deductions from them &c. But the
“first Principle of a Demonstration is an *immediate*
“Proposition, but that Proposition is *immediate*

† P. 132. Vol. I. of Du Val's Aristotle.

“which

6 A N E X A M I N A T I O N

“*which cannot be farther resolved,*” οὐ μὴ ἐστὶν ἄλλη πρό-
 “*τετα.* Aristotle does not give us a very accurate
 and satisfactory Idea of an immediate proposition,
 but so far he is perfectly clear, that no Proposition
 can be admitted as a first Principle of reasoning,
 which is capable of being demonstrated: and
 therefore if Demonstrations can be found of the
 Axioms of Geometry, they are not first Principles,
 and should not be taken for granted. In the 2d
 Case of Prop. 20. B. III. we are required to allow,
 that “if a Magnitude be double of another, and
 “a part taken from the first be double of a Part
 “taken from the second, the Remainder of the
 “first shall be double the Remainder of the se-
 “cond,” which is precisely a Case of Prop. 5. Book
 V. In the Doctrine of Proportion every thing is
 scientifically demonstrated; in the Substitutions and
 Transpositions of the Quantities of Equations, by
 the Addition and Subtraction of Magnitudes, every
 thing is taken for granted under the Name of
 Axiom. Shall we not then render an essential Ser-
 vice to the young Geometrician, if we can point
 out to him, to what Principle these Axioms may be
 referred?

The first and most simple Idea upon this Subject,
 (which we learn from Experience) is, that every
 Magnitude fills a certain Space; we find likewise
 from the most common Observation, that several
 Magnitudes may fill the same Space. These are
 Ideas which we cannot derive but from Observa-
 tion; and they are founded upon indubitable Facts,
 which are the only Basis of scientific Knowledge.

In this Manner we come by our Idea of *Equa-
 lity*; which is thus explained in Geometry, “Equal
 “Magnitudes are those, which coincide or fill the same
 “Space. Call this a self-evident Proposition, or
 what

what you please, it is really nothing more than a Definition of a Word; the Subject and Predicate of which both express the same thing in different Terms, and *therefore* admit no *middle Term*.

This Definition, which has generally been placed among the Axioms of Euclid, was inserted there by some later Geometrician. Euclid's Axioms, according to Proclus, are *common* Notions, ἡ τῆς γεωμετρικῆς ὅλης, "not peculiar to the Subject of Geometry." Now the Definition of Equality is purely geometrical; and herein consists the principal Advantage, which Geometry has above the other mathematical Sciences, whose Equations have not yet been explained in so accurate and scientific a Manner. Other Branches of Knowledge are still more remote from perfect Demonstration: Some very ingenious Men have indeed supposed, that these may be brought to the same perspicuity; but we do not find their hopes grounded upon any other Foundation than a common-place Idea, that no Limits can be prescribed to future Researches.

The Definition of Equality being thus confined to Geometry, we must endeavour to fix the Place, which it ought to hold among the Definitions. The Definitions of a Point, Line and Surface, which explain the Subject Matter of our Reasoning, must undoubtedly come first: and immediately after them, I think, that of Equality; For we find the straight Line, the plane Surface, the right Angle, the Circle, the equilateral, equiangular and isosceles Triangles, the regular four Sided Figures, in short, every specific Definition referred to this universal Principle; and with Regard to a few more general Definitions, which do not furnish an Equation, we shall find, that some Hypothesis is always made reducing them to that Principle, before any Theory
is

is built upon them. The first Example of which will be found in Prop. 4. Book I: and many more in the Course of the Elements. —That we may introduce as little Confusion as possible in the Arrangement of the Elements, let the Definition of Equality be called A; the original ones all retaining their former Numbers and Order.

It now only remains to shew, that this Definition is the Test or middle Term of the Axioms of Euclid: In which should we succeed, it will be allowed, that the first Principles in Geometry are not arbitrary Assumptions; and that Requisition of Aristotle will be fully answered, “that the first Principles should be better known than the Deductions from them.”

Axiom I. “Magnitudes, which are equal to one and the same, are equal to one another.”—Being equal to one and the same they fill the same Space (Def. A), therefore they are equal to one another. We find in Proclus, that Apollonius gave this same Demonstration of this Axiom: to which Proclus objects for two Reasons, first, because it is not manifest, that equal Things fill the same Space; and 2dly, that if they fill the same Space with one and the same, it does not follow that they fill the same Space with one another. I should be surprized to find so ingenious a Writer doubting, whether a given Space is equal to itself, were it not evident, that Proclus’s Objections arose from his too great partiality to Euclid, who never gave any Definition of Equality. We are told by Ramus in his *Proœmium Mathematicum* p. 406, that Regiomontanus, after the Example of Apollonius, demonstrated the Axioms of Euclid.

In

In the six following axioms are laid down the principles of the addition and subtraction of magnitudes, which, together with the preceding axiom, correspond with the rules of the most simple transpositions and substitutions in algebra. I will endeavour to demonstrate the 1st. of them, from which the rest easily follow.

Axiom 2. "If equal magnitudes be added to equal magnitudes, the wholes are equal."—Equal magnitudes fill the same space (def. A), and being added to equals magnitudes, which fill the same Space, the wholes fill the same Space, therefore they are equal (def. A). The next, viz. "If equal magnitudes be taken from equal magnitudes" &c. may be demonstrated almost in the same manner; as may also the two succeeding ones. The two last, namely, "Magnitudes, which are doubles of the same, are equal to one another;" and "Magnitudes which are halves of the same &c." are particular cases of 2. and 3. All the axioms in book V. are the additions of modern geometricians, and are contained in 2. and 3. of book I.

Axiom 9. "A whole is greater than it's part." The demonstration of this evidently follows from a definition of the word *greater*, which may be subjoined to that of equality, viz. *A greater magnitude is that, which fills more space than another magnitude.* Hobbes, Element. Philosoph. part. 2. cap. 8. has chosen this axiom as an example to prove, that Euclid's axioms are demonstrable.

Axiom 10. "Two straight lines cannot enclose a space." This is contained in the definition of straight lines.

Axiom 11. Which in some copies is postulate 4. "All right angles are equal to one another." This has been demonstrated by Proclus.

B

Axiom

Axiom 12. which is sometimes called postulate 5. "If a straight line meets two straight lines, so as to make the two interior angles on the same side of it taken together, less than two right angles, these two straight lines, being continually produced, shall at length meet on that side, on which the interior angles are less than two right angles."

"In pulcherrimo geometriæ corpore duo sunt nævi, duæ labes, nec quod sciam plures; prior est hoc postulatum;" these are the words of Savile in his 7 lecture. The Greek commentator expresses himself as follows, "It can with no degree of propriety be placed among the postulates; for it is a theorem admitting of many doubts, to solve which Ptolemy even undertook a book upon that subject; the demonstration of it requiring many definitions and theorems: and indeed Euclid himself demonstrates it's converse as a theorem. Some have thought proper to call it a postulate, because the diminution of the angles furnishes a *probability*, that the lines will meet; to which Geminus justly replied, that we are to pay no regard to probable appearances in considering geometrical proofs." Again, continues Proclus, supposing it evident, that the lines even approached to each other, it by no means follows, that they will meet, though it is probable; for there are lines, which are demonstrated to approach continually to each other, and never to meet, which is improbable, and even paradoxical. It is manifest therefore, that a demonstration of this should be sought, and that is has the property of a postulate." The same arguments prove likewise, that it cannot be an axiom, if

if by axiom be meant a self-evident proposition, as is now generally supposed.

Doctor Simson has attempted a demonstration of this proposition; in which, after two definitions, he assumes as an axiom, that "A straight line cannot first come nearer to another straight line, and then go further from it, before it cuts it; and, in like manner, a straight line cannot go further from another straight line, and then come nearer to it; nor can a straight line keep the same distance from another straight line, and then come nearer to it, or go further from it, for a straight line always keeps in the same direction." That a straight line always keeps in the same direction, is not a geometrical proof of the properties he assumes: what Proclus objected against the proposition itself, will with equal propriety apply to this axiom. — Clavius made the same attempt with no better success. Dechale has proposed a different axiom, which is as exceptionable as the original one.

In geometry, as in algebra, there is no reasoning but by equation: and some quantities evidently must be given, or supposed to be known, either absolutely, or in ratio, before an equation can be made. For a proof of this I refer to the elements. In the 4 first books there is no single proposition, whether it relates to the inclination of lines, construction of figures, or any property of them whatsoever, but it is proved, or done by the equality or inequality of certain lines, angles, or spaces. So likewise the whole theory of ratios; the geometrical principles of which are contained in def. 5. of the 5th book; which refers to the notions of greater, equal, and less for the test and measure of proportion; and thereupon de-

pende the whole doctrine. In treating of the definition of equality, I shewed, that all the specific definitions in the 1st book are determined by the same principle; and those general ones, which were expressed without it, must always be reduced to it by some hypothesis, before they become the subject of reasoning.

For the knowledge, we have of magnitude, is purely relative, and the most simple relation is that of equality. With this idea in common life we arbitrarily fix upon any magnitude, as a common measure, to which we refer other magnitudes by immediate application, and by which we express their quantity. Quantity therefore is nothing more, than the relation of magnitudes to some given thing, which is agreed upon, and thence supposed to be known

In science likewise, however our comparisons may be assisted and extended by the modification of space, they are all founded upon equality. An hypothesis or definition is taken as a test or common measure; other magnitudes are compared with that one, by shewing their possible or actual coincidence: and the only difference is, that in science we have certain contrivances to discover various equalities, or coincidences, without immediate application; and thereby to extend our researches.

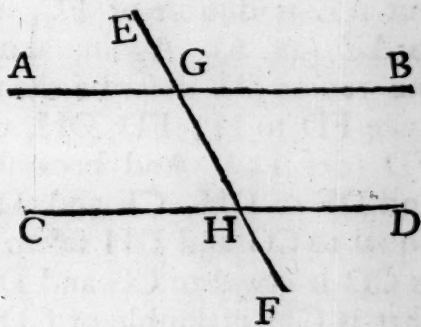
Those therefore, who attempted to investigate the properties of parallel lines from the original definition, failed, because they attempted an impossibility. They began with wrong, or rather without any, principles: that definition being only a vulgar and inaccurate conception; containing no given finite quantity, and furnishing no
measure

measure or standard, from which an equation might be made. What could be expected from such a definition in a science, all whose axioms express nothing but equations? — In geometry, properties of figures must be deduced from quantities, upon which our knowledge of them depends: unfortunately, in the doctrine of parallel lines, a property, namely, that they never meet, has been assumed, and from that many unsuccessful attempts have been made to discover quantities; as we find in the 29th prop. of the 1st book. What could not be demonstrated, was therefore taken for granted, and became the axiom, or postulate, of which we are now treating.

Hence it appears a priori, that the doctrine of parallel lines can never be made perfect, so long as the present definition remains. Take a proper definition, and you will find this subject as clear as any in the elements of geometry.

DEFINIT. B. Parallel straight lines are such, as are in the same plane, and upon which a straight line falling makes the alternate angles equal.

Let the straight line EF fall upon the straight lines AB, CD; the angles AGH and GHD are called alternate angles; as also the angles BGH and GHC.



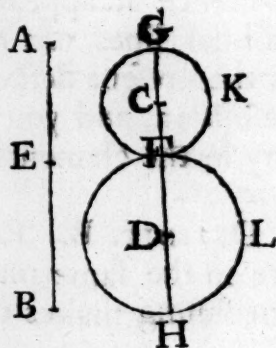
Here we have an equation given, from which the whole doctrine of parallel lines will be deduced. So in the definition of a circle, certain lines

lines are supposed equal to one another, and from them all the properties of the figure are demonstrated. The same may be said of every other geometrical theory.

But, I believe, none of the express axioms of Euclid will be thought more to require a demonstration, than the assumption in book III. above-mentioned, which has hitherto been tacitly acquiesced in. viz. "If a magnitude be double of another, and a part taken from the first be double of a part taken from the second, the remainder of the first shall be double the remainder of the second."

Let AB be double of CD, and let AE, a part of AB, be double of CF, a part of CD, EB, the remainder of AB, shall be double of FD, the remainder of CD.

About the center C, with the radius CF, describe the circle FKG; produce FC to G; FC and CG, that is, FG is double of FC (def. 15); but AE is double of FC, therefore FG is equal to AE (ax. 6). Again, about the center D, with the radius DF describe the circle FLH, and produce FD to H; FD, DH, that is, FH is double of FD (def. 15.) And because CF is equal to CG, and DF to DH, CF and DF taken together are equal to CG and DH taken together (ax. 2); that is CD is equal to CG and DH, and CD, CG, DH, that is GH is double of CD. But AB is double of CD; therefore GH is equal to AB (ax. 6). And GF has been proved to be equal to AE, therefore FH shall be equal to EB (ax. 3); but FH has been



been proved to be double of FD, therefore EB shall be double of FD. — Therefore, “If a magnitude be double of another” &c. q. e. d.

Thus we have shewn the origin and certainty of the genuine axioms; and deduced them all, as applied to geometry, from one definition, which is the universal principle of geometrical reasoning. Otherwise, this science resting upon several propositions respecting quantity, whose consistency with, and dependence upon, each other, is not shewn, wants at it's very foundation, that simplicity, harmony and compactness, which alone can give elegance and stability to the whole system.

PROPO-

PROPOSITIONS.

PROP. 4. In this and all other propositions, where the 10th axiom is quoted, the definition of a straight line, as above explained, fully answers the same purpose.

Corollaries. — Without corollaries, the chain of reasoning throughout the elements is complete and uninterrupted. By examining the corollaries as we go on, we shall find, that it is more than probable, that Euclid was not the author of any of them. But in a matter of this nature, in which the easy acquirement of an useful science is concerned, we should not be guided altogether by authority; if corollaries are found superfluous, and incompatible with the nature of the work, men of science will require no better reason for rejecting them; though it should not clearly appear, who was their author.

It is allowed, that Euclid wrote a book entitled, *πρὸς ἑσπεραν* “corollaries;” being a collection of consequences deducible from his elements. It is by no means probable, that the same author, who wrote a separate treatise of the corollaries, which followed from his elements, should insert in the elements, and under the same name, propositions not elementary, nor essential to that work. “From “a work of this nature,” says † Proclus,” all superfluity should be rejected, for that is an obstacle to knowledge; and it should contain nothing, but what is connected, and conducive to “the end proposed.” In the preceding page the

† P. 23.

same author justly observes, that "Euclid said not every thing, which might be said, but what was consistent with elements." — And therefore Euclid has not been more commended by some, for what he wrote, than for what he did not write. We cannot give the same praise to most of his commentators, who have been fond of inserting in the elements theories of their own, forgetting, that brevity is of the first importance in a series of demonstrations.

Coroll. to prop. 5. "Hence every equilateral triangle is also equiangular."

Coroll. to prop. 6. "Hence every equiangular triangle is also equilateral."

The two corollaries, which we are examining, have no more pretensions to originality, than they have to utility. They are not in Proclus, Grynæus, Gregory, Campanus's Translation from the Arabic, Zambertus, Tartalea, or Commandine. As to their signification, they are a part of their respective propositions; with as much propriety, we might subjoin to the definitions of equilateral and isosceles triangles, "hence every equilateral triangle is also isosceles," in short we might add any theory that occurs, and this compendious and beautiful system might be swoln into an enormous volume. Numberless corollaries might be deduced from every proposition: but, as the author of the Enquiry into the Sublime and Beautiful justly observes, "it is not the extent of the subject, which must prescribe our bounds, for what subject does not branch out to infinity? it is the nature of our particular scheme, and the single point of view in which we consider it, which ought to put a stop to our researches." — The evidence of any

C

theory

theory is not alone a sufficient reason for it's being inserted in a book, which is intended to furnish the mind only with principles, which the more compendious are the sooner learned and the more easily remembered. The design of elements is to abridge labour, by containing nothing, but what is necessary to our progress: unnecessary propositions therefore however evident have no place in them. Besides these advantages to the learner, it will likewise contribute to the beauty of this science, that the chain of demonstrations, which leads to it's sublimer truths, should be so continued, that their connection might be as obvious and comprehensible as possible. — The corollaries in general, which we find in the elements, are so incorrectly expressed and demonstrated, that they have afforded much trouble and controversy to commentators; therefore no great entertainment will be lost by rejecting them. However should any future editor be attached to some of them, let him at least remove them from the text; let them not interfere with the elementary propositions, but be inserted under the form of illustrations, notes, or commentaries.

Cor. to prop. 15. "All the angles made by any number of lines meeting in one point, are together equal to four right angles." Proclus mentions this, but does not ascribe it to Euclid. "A book of problems called corollaries, he says, Euclid wrote, but they are different from those in the elements." This corollary is not in Grynaeus, though he confesses that he found it in one copy. In Campanus it is made a part of the theorem. Zambertus has it not; nor Tartalea. Is it not evident therefore, that it is not essential to the elements, but has been arbitrarily adopted by any commentator, who happened to approve of it?

Prop.

Prop. 32. "If a side of any triangle be produced, the exterior angle is equal to the two interior and opposite angles; and the three interior angles of every triangle are equal to two right angles."

It were certainly to be wished, that this proposition, which relates only to the triangle, should be demonstrated, before we leave that subject. In Euclid it interferes with the theory of parallel lines, because it could not from his elements be demonstrated, without the assistance of the 29. and 31. propositions, or at least the former of these. But as the property of parallel lines referred to in the 29. prop. is contained in definition B, we are enabled to arrange proposition 32. in it's proper place; whereby we shall not only reform the order of the propositions, but considerably lessen the number of them; as will be seen hereafter.

For both these purposes the most convenient place for prop. 32. is immediately after prop. 15; where it may be demonstrated as follows;

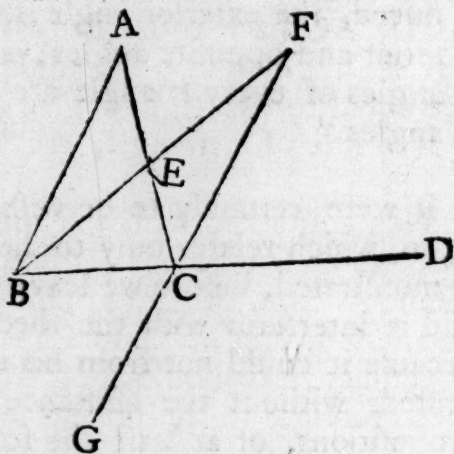
Prop. 32. Theorem. "If a side of any triangle be produced, the exterior angle is equal to the two interior and opposite angles; and the three interior angles of every triangle are equal to two right angles."

C 2

Let

Let ABC be a triangle, and let it's side BC be produced to D , the exterior angle ACD is equal to both the interior and opposite angles, CBA , BAC .

Bisect AC in E , join BE , and produce it to F ; and make EF equal to BE ; join also FC and produce it to G .



Because AE is equal to EC , and BE to EF ; AE , EB are equal to CE , EF , each to each; and the angle AEB is equal to the angle CEF , because they are vertical angles (prop. 15. B. I.); therefore the remaining angles in one triangle are equal to the remaining angles in the other, each to each, to which the equal sides are opposite (prop. 4. B. I.); wherefore the angle BAE is equal to the angle ECF ; but they are alternate angles, and when a straight line, falling upon two other straight lines, makes the alternate angles equal, those two straight lines are parallel (def. B); therefore AB is parallel to FC : and the straight line BC falls upon them, therefore the two alternate angles ABC , BCG are equal (def. B); but BCG is equal to FCD , because they are vertical angles (prop. 15. B. I.), therefore the angle FCD is equal to the angle ABC : and the angle ACF has been proved to be equal to the angle CAB , therefore the two angles ACF , FCD , that is, ACD the exterior angle, is equal to CAB , ABC , the two interior and opposite angles (ax. 2.); to each of these equals add the angle ACB ;

ACB; and the angles ACD, ACB are equal to the three angles CBA, BAC, ACB; but the angles ACD, ACB are equal to two right angles (prop. 13. B. I.), therefore the angles CBA, BAC, ACB are equal to two right angles (ax. 1.). Wherefore, "If a side of a triangle, &c." q. e. d.

Not contented with the many useful consequences, which Euclid has deduced in the course of his elements from this important proposition, his commentators, and translators have employed all their ingenuity in demonstrating from it various corollaries, foreign to the present enquiry, and of little or no use in geometry. Among several other corollaries, we find in almost every edition the two following;

Coroll. 1. "All the interior angles of any rectilineal figure whatever, together with four right angles, are equal to twice as many right angles as the figure has sides."

Coroll. 2. "All the exterior angles of any rectilineal figure whatever, taken together, are equal to four right angles."

I believe these corollaries, will be found at least as intricate as the generality of propositions, and are no more entitled to the name of *corollary* than any of Euclid's theorems, from which they differ only in being less useful.

Prop. 1. Theorem. "If one side of a triangle be produced, the exterior angle is greater than either of the interior and opposite angles."

Prop.

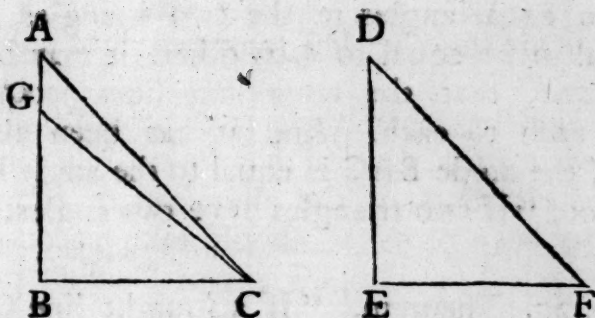
Prop. 17. Theorem. "Any two angles of a triangle are together less than two right angles."

It is evident, that these two propositions are contained in prop. 32. and may be entirely remov'd from the elements, if the alteration of prop. 32. which has been proposed, should take place. —But when we reject or transpose any of Euclid's propositions, which have been always quoted according to their present order, care must be taken to avoid as much as possible the inconveniences, which would be occasioned by such alterations. Therefore prop. 32. may be called *proposition A, formerly 32.* and in it's former place, a reference may be made to it's present situation.

Prop. 26. Theorem. "If two triangles have two angles of one equal to two angles of the other, each to each; and one side equal to one side, viz. either the sides adjacent to the equal angles, or the sides opposite to equal angles, in each; then shall the other sides be equal, each to each, and also the third angle of the one to the third angle of the other."

Let there be two triangles ABC, DEF, having the two angles ABC, BCA, of one triangle, equal to the two angles DEF, EFD, of the other, each to each; and also one side equal to one side, and first let BC be equal to EF, which are between the equal angles, in the two triangles; the other sides shall be equal, each to each, viz. AB to DE, and AC to DF, and the third angle BAC to the third angle EDF.

The



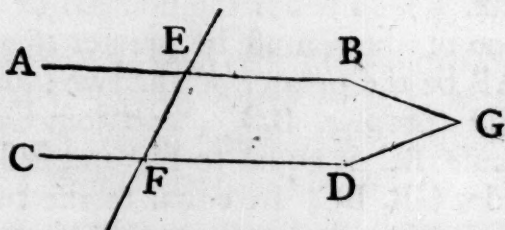
The three angles ABC , BCA , CAB , of the triangles ABC are equal to two right angles, (prop. A), as also the three angles DEF , EFD , FDE of the triangle DEF : therefore the angles ABC , BCA , CAB , taken together are equal to the angles DEF , EFD , FDE taken together (ax. 1.), but the two angles ABC , BCA , are equal to the two angles DEF , EFD (by supp.), therefore the remaining angle BAC , is equal to the remaining angle EDF (ax. 3.). Then if the side AB be not equal to ED , one of them must be greater than the other. Let AB be the greater of the two; make BG equal to ED (prop. 3. B. I.); and join GC ; therefore because BC is equal to EF and BG to ED , the two sides GB , BC , are equal to the two DE , EF , each to each, and the angle GBC is equal to the angle DEF , therefore the other angles are equal to the other angles, each to each, to which the equal sides are opposite (prop. 4. B. I.); therefore the angle GCB is equal to the angle DFE ; but the angle ACB is equal to the angle DFE , therefore GCB is equal to ACB (ax. 1.), the less to the greater, which is impossible; therefore AB is not unequal to DE , that is, it is equal to it; and BC is equal to EF (by supp.), therefore the two AB , BC , are equal to the two DE , EF , each to each, and the angle ABC is equal to the angle DEF ; therefore the base AC is equal to the base DF (prop. 4. B. I.). In the same

same manner, if either of the sides, which are opposite to equal angles in the two triangles, were supposed to be equal to each other, it may be demonstrated, that the remaining sides would be equal, each to each. And it has been already proved, the angle BAC is equal to the angle EDF. Therefore "If two triangles have two angles, &c." q. e. d.

Prop. 27. Theorem. "If a straight line falling upon two other straight lines makes the alternate angles equal to one another, these two straight lines shall be parallel." — Here begins the theory of parallel lines. Omitting therefore this proposition, which is definition B, I will demonstrate that parallel straight lines never meet, which is Euclid's definition.

Prop. A. Theorem. Parallel straight lines never meet.

Let AB, CD, be two parallel straight lines; AB, CD, shall never meet.



For suppose the lines AB, CD, to meet in any point, as G; let the straight line EF fall upon them, EFG, is a triangle; therefore the angle AEF is equal to the two internal and opposite angles, EFG, FGE (prop. A); therefore AEF is greater than EFG (ax. 9.); but a straight line falling upon two parallel straight lines makes the alternate angles equal, therefore AEF is equal to EFG (def. B); but it is also greater than it, which is impossible. Therefore AB, CD do not meet in G; and in the manner it may be demonstrated, that they do not meet towards A and C. Therefore, "Parallel straight lines, &c." q. e. d.

Prop.

Prop. 28. Theorem. "If a straight line falling
 "upon two other straight lines makes the exterior
 "angle equal to the interior and opposite one up-
 "on the same side of the line; or makes the inte-
 "rior angles upon the same side taken together
 "equal to two right angles, the two straight lines
 "are parallel."

This may be demonstrated from my definition of parallel straight lines as was done before from Prop. 27.

Prop. 29. Theorem. "If a straight line falls
 "upon two parallel straight lines, it makes the al-
 "ternate angles equal to one another; and the ex-
 "terior angle equal to the interior and opposite
 "upon the same side; and the two interior angles
 "upon the same side taken together equal to two
 "right angles."

We may omit that part of this proposition, which relates to the equality of the alternate angles, because it is the converse of our definition of parallel lines. A definition always implies its converse. If all the radii are equal, the figure is a circle, and if it be a circle, all the radii are equal. So likewise in the former doctrine of parallel lines, if right lines in a plane never met, they were parallel; and if parallel, they would never meet. Without this extent to our definitions we should lose half of our theories, as we should have no principle, by which the converse of our definitions could be demonstrated. The reasonableness of this rule will be acknowledged, when we consider, that a definition implies the very essence of the thing defined. The word parallel has no signification but what the definition gives it; the equality of the alternate an-
 D gles

gles is the only idea we can affix to it. The equality of the alternate angles is the thing, and parallelism a technical term to express it. The definition therefore, and the word defined, being one and the same thing, are with great propriety always taken conversely.

The demonstration of the 29. as it now stands, may be made out nearly as in the original.

Prop. 30. Theorem. "Straight lines which are
"parallel to the same straight line, are parallel to
"one another."

This may remain in it it's former state : as may also ;

Prop. 31. Problem. "To draw a straight line
"through a given point parallel to a given straight
"line."

Prop. 32. Theorem. "If a side of any triangle
"be produced, the exterior angle is equal to the
"two interior and opposite angles ; and the three
"interior angles of every triangle are equal to two
"right angles."

This is placed immediately after prop. 15. for reasons already assigned.

Prop. 33. Theorem. "The straight lines, which
"join the extremities of two equal and parallel
"straight lines towards the same parts, are also
"themselves equal and parallel."

The original demonstration of this requires no alteration.

I am,

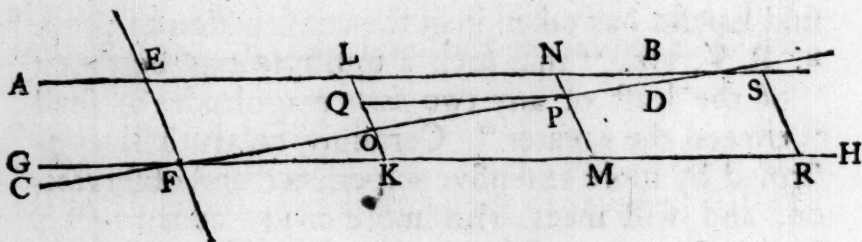
I am under some apprehensions least the subsequent demonstration may be thought to depend too much upon a particular interpretation of the word *finite*. If this word had been defined by Euclid, I should have endeavoured to refer my demonstration to his definition. But as this term is used in some places without any precise signification, as in prop. 1. and 10. B. I. and as it is supposed throughout the elements, that any *finite* magnitude may be increased, till it becomes greater than any other *finite* magnitude; I have taken the same liberty with this word, and have employed it in the same sense, in which I find Euclid has taken it in the construction of prop. 8. B. V. viz. "that such a multiple may be taken of the least of any two *finite* magnitudes as shall exceed the greater." Certainly no truth is confirmed by more extensive experience and observation, and will meet with more ready concurrence. Yet it is not consistent with the rules of an abstracted science to reason upon terms, which have not been defined. The want of a proper definition of the word *finite* has, I think, been the occasion of the following inaccurate definition being inserted in B. V. "Magnitudes are said to have a ratio to one another, when the less can be *multiplied* so as to exceed the greater;" of which more will be said hereafter. All these inconveniencies may be totally removed by inserting the following among the definitions of the first book.

Def. C. A magnitude is said to be finite, when any magnitude whatever may be taken such a number of times as to be greater than it.

Prop. B. Theorem: formerly Postulate 5. or Axiom 12. "If a straight line, falling upon two straight lines, makes the two interior angles
D 2 "upon

“ upon the same side of it less than two right angles, these straight lines being continually produced shall at length meet on that side, on which the two interior angles are less than two right angles.”

Let the straight line EF falling upon the two straight lines AB, CD, make the two interior angles on the same side of it, viz. BEF, EFD, less than two right angles, the two straight lines AB, CD, if continually produced towards B and D shall at length meet.



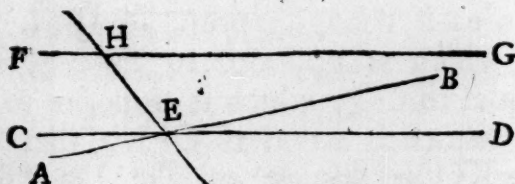
Through the point F draw the straight line GH parallel to AB (prop. 31. B. I.); and because the straight line EF falls upon the parallel lines AB, GH, it makes the two angles BEF, EFH, equal to two right angles (prop. 29. B. I.): but the two angles BEF, EFD, are less than two right angles, therefore the two angles BEF, EFH, are greater than the two angles BEF, EFD, take from each the common angle BEF, and there will remain EFH greater than EFD, therefore the the straight line FD falls within the angle EFH. In the straight line FH let any point K be taken, and make EL in the straight line AB equal to FK (prop. 3. B. I.), and join KL; KL is parallel to EF (prop. 33. B. I.); and KL shall meet the straight line CD, unless CD shall have passed the line AB, in which case, what was required is already demonstrated; but if
not

not, KL will meet it in some point as O. In the straight line FH make KM equal to FK, and also LN in the straight line AB equal to KM (prop. 3. B. I.): and join MN; the straight line MN is parallel to KL and FE (prop. 33. B. I.); and it may be demonstrated, that the straight line MN shall meet the straight line CD, unless CD shall have passed AB, in the same manner as it has been already demonstrated with respect to KL; let MN meet CD in any point at P: in the straight line KL suppose KQ to be made equal to MP: and join PQ; the straight line PQ is parallel and equal to KM (prop. 33. B. I.); and the straight line LK falls upon them, therefore the angle FKQ is equal to the angle KQP, because they are alternate angles (def. B.); and for the same reason, because the straight line CD falls upon the two parallel lines QP, FM, the angle MFP shall be equal to the angle FPQ, therefore in the two triangles POQ, FOK, we have two angles in one equal to two angles in the other, each to each; and the sides FK, PQ, which are adjacent to the equal angles, are equal to each other, for they are each of them equal to KM; therefore the remaining sides are equal, each to each, in each triangle (prop. 26. B. I.), therefore OK, which is opposite to the angle OFK, shall be equal to OQ, which is opposite to the angle OPQ, which is equal to OFK: therefore the whole line KQ is double of the Line KO, but KQ is equal to MP (prop. 33. B. I.), therefore MP is double of KO. In the same manner if MR be made equal to KM, and NS equal to MR, it might be demonstrated that if RS meet CD, the part of it intersected between the straight line GH and CD, would be equal to MP and KO taken together; and so on continually; but any magnitude
may

may be taken such a number of times as to be greater than any other finite magnitude (def. C.); therefore if OK be continually added to PM it shall at length become greater than MN; that is, than RS which is equal to MN (prop. 33. B. I.); but, when this is the case, the straight line RS which joins the parallel straight lines AB, GH, does not meet the straight line CD; therefore CD must have passed the line AB. Therefore "If a straight line, "falling upon two other straight lines, makes the "two interior angles upon the same side less than "two right angles, these two straight lines being "continually produced shall at length meet on that "side on which the two interior angles are less than "two right angles." q. e. d.

Prop. C. Theorem. If a straight line meets another straight line, it shall, if continually produced, meet any straight line which is parallel to it.

. Let the straight line AB meet the straight line CD in the point E; and let FG be parallel to CD, AB shall, if produced, meet FG.



From the point E, let any straight line as EH be drawn, meeting FG in any point. as H; the two interior angles GHE, HED taken together are equal to two right angles (prop. 29. B. I.) but HEB is less than HED, therefore the two angles GHE, HEB, taken together, are less than two
right

right angles ; but when a straight line falling upon two other straight lines makes the two interior angles upon the same side less than two right angles, these two straight lines being continually produced shall meet on that side on which the two interior angles are less than two right angles ; therefore AB shall, if continually produced, meet FG. Therefore, If a straight line meets, &c. q. e. d.

The Utility of this proposition, which has hitherto been taken for granted, will be obvious to every person, who has read a book of Euclid's Elements.

Prop. D. Theorem. Straight lines, which being continually produced never meet, are parallel.

Let AB, CD be two straight lines, which being continually produced never meet, AB shall be parallel to CD.



For suppose they are not parallel ; let fall upon them any straight line EF ; the angle AEF is not equal to the angle EFD, for if it is the two lines AB, CD are parallel, and what was required is already demonstrated (def. B.) : therefore one of them is greater than the other ; suppose AEF to be greater than EFD, to each of them add FEB : AEF, FEB are equal to two right angles (prop.

(prop. 13. B. I.), therefore DFE, FEB are less than two right angles; but when a straight line falling upon two other straight lines makes the two interior angles upon the same side less than two right angles, these straight lines shall, if continually produced, meet on that side, on which the two interior angles are less than two right angles (prop. B.); therefore EB, FD being produced shall meet; but by supposition they never meet: therefore AEF is not greater than EFD, and in the same manner we might prove that it is not less than it; therefore it is equal to it: but AEF, EFD are alternate angles, therefore AB is parallel to CD (def. B.). Therefore Straight lines, &c. q. e. d.

Prop. 46. Problem. "To describe a square upon a given straight line."

Euclid gave no definition of the rectangle, probably because the Greek expression, *παρλληλογραμμον ορθογωνιον*, or *ορθογωνιον* alone, *παρλληλογραμμον* being understood, is a definition of the figure. In the English language *rectangle* has no signification, and therefore it ought to be defined before we treat of it. It was not necessary, nor indeed possible, to define this figure, before some properties of parallel lines were demonstrated; and the attempts, which have been made to give definitions of the *oblong* and *square*, before the propositions, have been observed to be imperfect and unsuccessful.

Def. D. A rectangle is a parallelogram having all its angles right angles.

Def. E. A square is a rectangle having all its sides equal.

This

This definition of a square, by referring that figure to its proper species, renders it unnecessary to treat of it separately, as is done in the proposition, which we are now examining. Besides, after prop. 42. 44. and 45. it will be needless to demonstrate how a rectangle may be constructed when its adjacent sides are given, or, which is the same thing, how to construct a square upon a given straight line.

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EXAMINATION
OF THE
FIRST SIX BOOKS
OF
EUCLID'S ELEMENTS.
BOOK II.

PROPOSITIONS.

PROP. 1. Theorem. "If there be two straight
" lines one of which is divided into any num-
" ber of parts; the rectangle contained by the two
" straight lines is equal to the rectangles contained
" by the undivided line, and the several parts of
" the divided line."

Prop. 2. Theorem. "If a straight line be di-
" vided into any two parts, the rectangles contain-
" ed by the whole line and each of the parts are to-
" gether equal to the square of the whole line."

Prop. 3. Theorem. "If a straight line be di-
" vided into any two parts, the rectangle con-
" tained

“tained by the whole line and one of the parts
 “shall be equal to the rectangle contained by
 “the two parts together with the square of the
 “foresaid part.”

The two last mentioned propositions do not seem on account of any great utility they are of, to be entitled to more than one demonstration; and I think the demonstration of them both is comprehended under that of the first. For whatever is demonstrated of the rectangle in general, is likewise demonstrated of the square; which is an equilateral rectangle. In the first proposition it is demonstrated, that the rectangle contained under any two lines whatever, shall be equal to all the rectangles contained under the several parts, into which one of them is divided, and the other undivided line. Is it any exception to this proposition, if the two right lines are equal to each other? in which case we have prop. 2. as a square is a rectangle contained under two equal lines.—Prop. 3. is another case of the first; in which the “undivided line” is equal to one of the segments of the divided “line,” which is the only difference between this and the first. I have often observed that such propositions perplex learners more than those of real importance; and occasion an unnecessary delay in their progress to the more sublime and entertaining parts of mathematical learning. These superfluities and some others of the same nature, have crept into the elements, from the square being considered as a different figure from the rectangle; and not classed, as it ought to be, under the same species.

Cor. to prop. 4. “From this it is manifest, that
 “the parallelograms about the diameter of a square
 “are likewise squares.”

The

The fourth proposition has two demonstrations in the original; which is not uncommon; and when this is the case, a corollary generally follows; which it is probable was inserted by the author of the second demonstration.

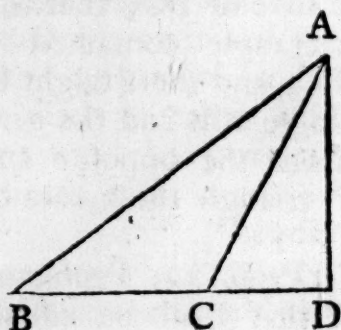
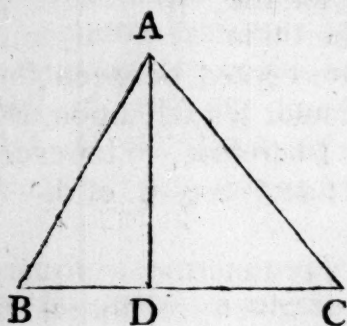
Prop. 13. Theorem. "Εν τοῖς ὀξυγωνίοις τρίγωνοις,"
 " &c. In acute angled triangles the square of
 " the side subtending any of the acute angles,
 " is less than the squares of the sides containing
 " that angle, by twice the rectangle contained
 " by either of these sides, and the straight line
 " intercepted between the perpendicular let fall
 " upon it from the opposite angle, and the acute
 " angle."

In the *Εκθεσις* of this proposition we have, in the Greek, "let ABG be an acute angled triangle
 " having the angle at B *acute*," which is mere tautology. Besides, it has been justly observed that this theorem is not confined to acute angled triangles, but holds true of acute angles in any triangle. And therefore Simson and some others have given two more cases with different demonstrations to supply the defect. A little attention would have shewn them, that with proper diagrams the original demonstration is applicable to two of the different cases.

Prop. 13. Theorem. In every triangle the square of the side subtending any of the acute angles, is less than the squares of the sides containing that angle, by twice the rectangle contained by either of these sides, and the straight line intercepted between the perpendicular let fall from the opposite angle upon it, produced if necessary, and the acute angle.

Let

Let BAC be any triangle, and the angle at B one of it's acute angles, and if AC be not perpendicular to BC , let fall upon BC , produced if necessary, the perpendicular AD from the opposite angle: (prop. 12. B. I.) The square of AC , opposite to the angle B , is less than the squares of CB , BA , by twice the rectangle CB , BD .



The squares of CB , BD , are equal to twice the rectangle contained by CB , BD , together with the square of DC (prop. 7. B. II). To each of these add the square of AD ; and the squares of CB , BD , DA , are equal to twice the rectangle CB , BD , and the squares of AD , DC . But the square of AB is equal to the squares of BD , DA , because the angle BDA is a right angle (prop. 47. B. I.); and the square of AC is equal to the squares of AD , and DC : therefore the squares of CB , BA , are equal to the square of AC , and twice the rectangle CB , BD ; that is, the square of AC alone is less than the squares of CB , BA by twice the rectangle CB , BD .

But

But if BAC be a right angled triangle, the perpendicular let fall from A coincides with the side AC. In this case the square of AB is equal to the squares of AC, CB, (prop. 47. B. I.); therefore the square of AC is less than the square of AB, BC, together by twice the square of BC; that is, by twice the rectangle contained by the side BC, and the straight line intercepted between the angle at B and the perpendicular let fall upon BC from the opposite angle. Therefore, "In every triangle the square of the side," &c. q. e. d.



Prop. 14. Problem. "To describe a square that shall be equal to a given rectilineal figure."

In the original construction of this a very injudicious step was taken, which Simson has corrected.

When we observe, that almost every deviation from the subject in hand is attended with some inaccuracies, and that the leading propositions in these elements are the most correct and perfect that occur in any work, we shall think it at least probable, that those propositions only are Euclid's which carry us forward in the prosecution of our enquiries, and that those, which interrupt the chain of reasoning, have crept in from the imprudent zeal or vanity of his successors, to enrich elements with theories of their own. The two last propositions of the first book, or the three last if we admit prop. 46, are indeed a separate part; but they are absolutely necessary; neither could they

40 AN EXAMINATION, &c.

they be demonstrated conveniently in any other place, as several propositions in the second book could not be solv'd without the assistance of prop. 47. B. I. The three last propositions of B. II. are a detach'd part, but not absolutely necessary in the elements; and I believe of no material utility in any other place: particularly the two former of them. With respect to the last it is no where quoted in the first six books; and no person will want a demonstration of it after reading prop. 13. and 17. B. VI.

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OF THE
FIRST SIX BOOKS
OF
EUCLID'S ELEMENTS.

BOOK III.

PROPOSITIONS.

COR. to prop. 1. "From this it is manifest,
"that if in a circle a straight line bisect ano-
"ther at right angles, the center of the circle is
"in the line which bisects the other."

This corollary is fully contained in the proposition, or rather is the very principle, by which the problem is solv'd.

Prop. 6. Theorem. "If two circles touch each
"other internally, they shall not have the same
"center."

This theorem *evidently* holds true of external as well as internal contact. It is certainly an unnecessary

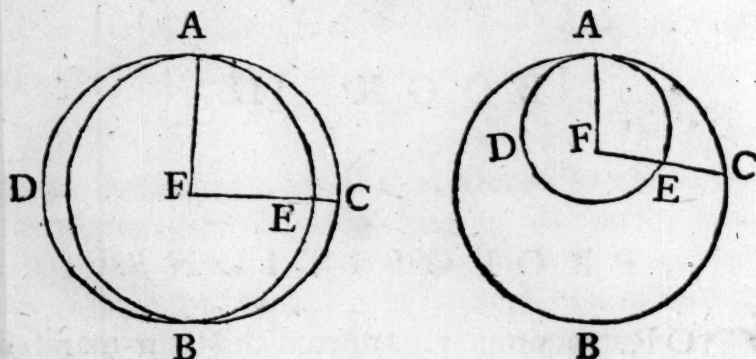
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fary precaution to restrict any theorem, and to exclude one of its cases merely on account of its evidence, when it comes under the general demonstration. By giving the enunciation of this proposition without any limitation, we obtain the advantage of uniting prop. 5. and 6. without altering a word of their demonstration.

Prop. 5. and 6. If two circles cut or touch each other they shall not have the same center.

Let the two circles ABC, ADE cut each other in the points A, B, or touch each other in the point A, they shall not have the same center.



For, if it be possible, let F be their center, join FA and draw any straight line meeting them in E and C: because F is the center of the circle ABE, FE is equal to FA; and because F is the center of the circle ABC, AF is equal to FC, but AF has also been proved to be equal to FE, therefore FE is equal to FC, the less to the greater; which is impossible. Therefore F is not the center of both the circles ABG, ACE.

Therefore, If two circles cut, or touch each other, &c. q. e. d.

Prop.

Prop. 7. Theorem. "If any point be taken in
 "the diameter of a circle, which is not the center,
 "of all the straight lines which can be drawn from
 "it to the circumference, the greatest is that
 "in which the center is, and the other part of that
 "diameter is the least; and of any others, that
 "which is nearest to the line which passes
 "through the center is always greater than one
 "more remote: And from the same point there
 "can be drawn only two straight lines that are
 "equal to one another, one upon each side of
 "the shortest line."

In the last case of this and the following proposition the line passing through the center of the circle is called the *shortest line*.

Prop. 8. Theorem. "If any point be taken
 "without a circle, and straight lines be drawn
 "from it to the circumference, whereof one passes
 "through the center; of all those which fall up-
 "on the concave circumference, the greatest is
 "that which passes through the center; and of the
 "rest, that which is nearer to that through the cen-
 "ter is always greater than the more remote: But
 "of those that fall upon the convex circumference,
 "the least is that between the point without the
 "circle and the diameter, and of the rest, that
 "which is nearer to the least is always less than the
 "more remote: And only two equal straight lines
 "can be drawn from the point to the circumfe-
 "rence, one upon each side of the least."

From the first of these propositions follows so easy and natural a demonstration of prop. 9. that we cannot suppose Euclid could have overlooked it.

The ninth, namely ; “ If a point be taken within a circle, from which fall more than two equal straight lines to the circumference, that point is the center of the circle,” is the converse of the last case of the seventh. Some later geometrician has deduced this from the seventh, in the second demonstration, which is generally preferred to Euclid's. Now as the seventh proposition is used no where but in this demonstration, and here not by Euclid, unless we suppose Euclid gave two demonstrations of the same proposition, and as the eighth is no where used, I believe, throughout the Elements, it is not improbable that they were both invented by the author of the second demonstration of the ninth proposition, who alone seems to have known their use.

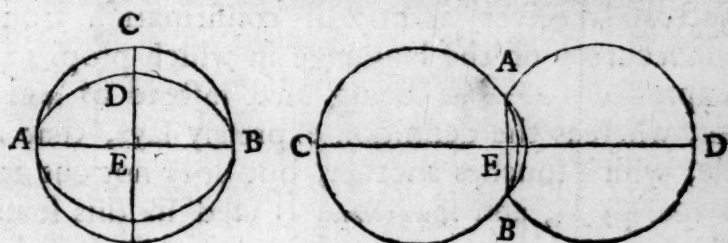
For the theory of the circle is complete without them. In prop. 14. it is demonstrated that lines equidistant from the center are equal, and the converse of this. The next proves that of all the right lines in a circle the diameter is the greatest ; and that the one nearer to the center is always greater than one more remote ; which is a general comparison of all lines terminating both ways in the circumference of a circle, from their situation respecting the center. But I think even this might well be omitted. For these relations are more exactly ascertained in prop. 45. of this book ; which proposition completes the theory of the circle, and furnishes those principles upon which the equation to the curve is established.

The relative magnitudes of the segments of lines drawn from a point without the circle, to the concave and convex circumference, are determined in proposition 46.

To

To this it may be objected, that Theodosius in his Spherics refers to these propositions and demonstrates similar properties of the sphere. But, according to Montucla, the author of the spherics liv'd 300 years after Euclid, too distant a period to prove the originality of any part of this work. Neither is this, or any other reference, an argument for retaining unnecessary propositions in the elements, which are not to be consider'd as a common place book, but as a system.

Prop. 13. Theorem. "One circle cannot touch another in more points than one, whether it touches on the inside or on the outside."



For, if it be possible, let the two circles ABC, ADB touch each other in the two points A and B. Join AB, and bisect the line AB, as in E; through the point E draw the straight line EC at right angles to AB. The straight line EC produced shall pass through the center of both the circles (prop. 1. B. III.); but the straight line which joins the centers circles that touch each other, either internally or externally, shall pass through the point of contact (prop. 11. and 12. B. III.) therefore CE produced shall pass through the point of contact; but it does not pass through it because the points A and B are not in the straight line CE; which is absurd. Therefore, "One circle cannot touch another in more points than one," &c. q. e. d.

Thus

Thus the two cases of this proposition may be comprehended under one demonstration: the second case of it following as obviously from the twelfth as the first does from the eleventh proposition of this book.

This singular oversight induces me to suspect that prop. 12. "If two circles touch each other externally, the straight line joining their centers, passes through the point of contact," was not in the original elements. I have before observed, that Euclid excludes circles which touch externally, from prop. 6. when he shews that those which touch internally have not the same center. This conjecture receives additional confirmation from the inaccuracy of the language in which prop. 12. is expressed; *απτανται* being used instead of *εφαπτανται*, whereas the definition expressly says, that a circle, which touches another, but does not cut it, *εφαπτεσθαι λεγεται*, and *εφαπτανται* is used in this sense in the very preceding proposition. Yet it must be owned that, in some other instances, the proper distinction is not preserved.

Prop. 16. Theorem. "The straight line drawn at right angles to the diameter of a circle, from the extremity of it, *falls without the circle*; and no straight line can be drawn between that straight line and the circumference, from the extremity of the diameter, so as not to cut the circle", &c.

By deviating a little from his usual mode of expression in the first case of this proposition, Euclid has given occasion to some person to subjoin a corollary to shew what is strictly demonstrated, but
in

in other words; viz. "that a straight line drawn at right angles to the diameter at its extremity *shall touch the circle.*" Therefore to conform with the definition of a tangent, let the proposition be expressed, as follows; The straight line drawn at right angles to the diameter of a circle, from its extremity, *does not cut the circle*; which may be demonstrated as the former.

Prop. 20. Theorem. "The angle at the center of a circle is double of the angle at the circumference, upon the same base, that is upon the same part of the circumference."

This proposition is general; and it is immaterial whether the part of the circumference, upon which the angles at the center and circumference stand, be greater or less than a semicircle. Simson and some other modern Editors of Euclid, who have made an addition of a new case to the following proposition, seem not to have observed the full extent of this.

Prop. 21. Theorem. "The angles in the same segment of a circle are equal to one another."

Any angle at the circumference of a given segment is half of one and the same angle at the center, as was shewn in the preceding proposition, whether the segment be greater or less than a semicircle. Yet, as was observ'd above, another demonstration has been given of late, to shew that the angles in a segment less than a semicircle are equal to each other.

Prop. 25. Problem. "A segment of a circle being given to find the circle of which it is a segment."

To

To the three cases of this proposition, one of which Simson has shewn to be unnecessary, are subjoined the following remarks upon the demonstration. "And it is evident, that the segment ABC "is less than a semicircle, because the center E "is without it"; "And evidently ABC is a semicircle"; "And evidently ABC is greater than a "semicircle". By supposition the segment of the circle is *given*; these observations seem to imply, that it is discover'd from the demonstration.

If from a segment of a circle being given the sides and angles of a triangle inscribed in it are known, which is supposed in the construction of this problem, this proposition might be omitted, being contain'd in that more general one in Book IV, "To describe a circle about a given "triangle."

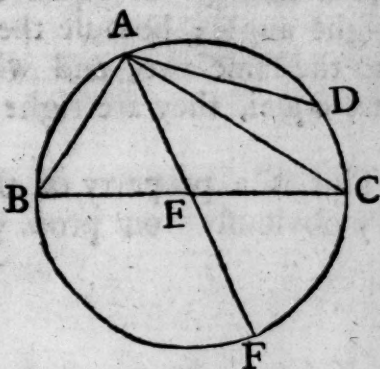
Prop. 31. Theorem. "In a circle the angle in "a semicircle is a right angle; but the angle in "a segment less than a semicircle is greater than a "right angle."

Angles at the circumference of a circle are most conveniently measured, by those at the center standing upon the same base: And this demonstration, if referred to that principle would be sooner understood and more easily remember'd, than the present one which is rather complicated.

Let BADC be a circle, A any point in its circumference, BC its diameter, and E its center. Join AB, AC; and draw the straight line AD to any point in the segment AC. The angle BAC in a semicircle is a right angle, and ABC in a segment greater than a semicircle is less than a right angle,

angle, and BAD in a segment less than a semicircle is greater than a right angle.

Draw the diameter AF. DEF the angle at the center is double of BAF the angle at the circumference (proposition 20. B. III); for the same reason FEC is double of FAC; therefore the two angles BEF, FEC are double of the two angles BAF, FAC, but BEF, FEC, are equal to two right angles (prop. 13. B. I.), therefore BAC is a right angle. Wherefore the angle in a semicircle is a right angle.



All the angles in the triangle BAC are equal to two right angles (prop. 32. B. I.), but BAC is a right angle, therefore ABC is less than a right angle. Therefore the angle in a segment greater than a semicircle is less than a right angle.

The angle BAD which is in a segment less than a semicircle is greater than the right angle BAC. Therefore the angle in a segment, less than a semicircle, is greater than a right angle. Therefore, "In a circle, the angle in a semicircle, &c." q. e. d.

In the former demonstration some observations were made upon the angles contained by an arc of a circle and its chord, after which comes the usual conclusion, "which was to be demonstrated," though nothing was proposed upon that subject: this therefore may be omitted; as may also the corollary, which is equally foreign to the subject.

Cor. " From this it is manifest that if one angle
" of a triangle be equal to the other two it is a
" right angle ; because the angle adjacent is equal
" to the same two, and when the adjacent angles
" are equal, they are right angles."

This is a property of the triangle, and follows
very obviously from prop. 32. B. I.

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BOOK IV.

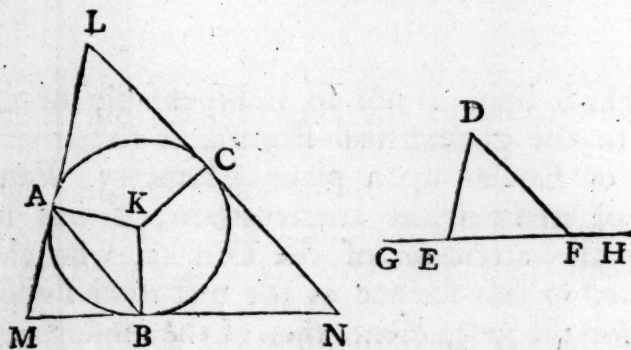
THIS book is not so indispensably necessary to the geometrical student as the other five books of Euclid upon plane geometry. Being a series of geometrical constructions, it was more worthy the attention of the Greeks, who chiefly delighted in this science as the best model and exercise for the judgement, than of the moderns, who have seen mathematical learning happily applied to various uses in philosophy, unknown to the Greeks. Besides, in this age, men in general have less time to employ in mathematical studies, than the ancients had: and many give up the undertaking before they have advanced far enough in this science to know the use of it. Our progress therefore should be, if possible, more rapid; and the elements of this science should be rendered easier, as its application is become more extensive, and

the system of education in general more complicated. With this view the fourth book might be pass'd over very cursorily, if not entirely omitted: and might be consulted occasionally, whenever these constructions are wanted, which very rarely happens.

PROPOSITIONS.

Prop. 3. Problem. "About a given circle to describe a triangle equiangular to a given triangle."

Let ABC , be the given circle, and DEF the given triangle, it is required to describe a triangle about the circle ABC , equiangular to the triangle DEF .



Produce EF both ways to the points G, H ; and find the center K of the circle ABC ; and from it draw any straight line KB : make the angle BKA equal to the angle DEG , and the angle BKC equal to the angle DFH (prop. 23. B. I.); and through the points A, B, C , draw the lines LAM, LCN, MBN , touching the circle in the points A, B, C (prop. 17. B. III.); join AB ; the two angles KBM, KAM are equal to two right angles (prop.

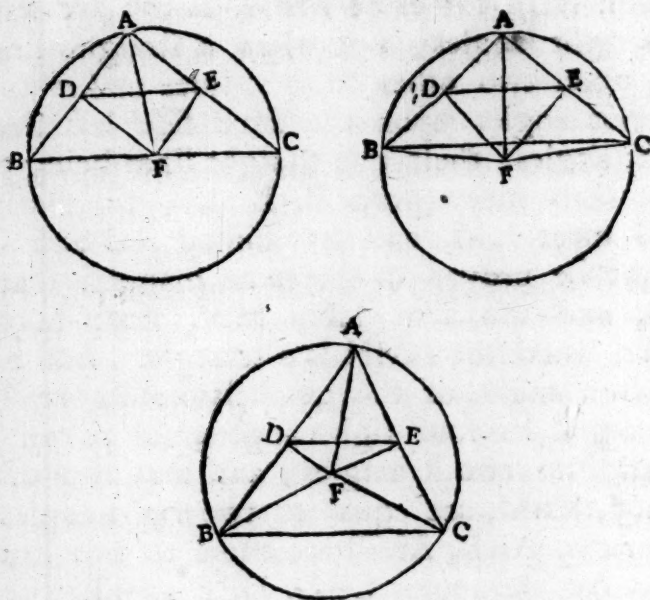
18. B. III.), therefore $\angle MBA$, $\angle BAM$ are less than two right angles; but when a straight line falling upon two other straight lines makes the two interior angles upon the same side less than two right angles, these two straight lines being produced, shall meet (prop. B. B. I.); therefore NM shall meet LM , and by joining AC and CB , it might be proved in the same manner, that ML , LN , and also that LN , MN , shall meet each other; therefore LMN is a triangle. And because the four angles of the quadrilateral figure $AKBM$ are equal to four right angles, for it can be divided into two triangles; and that two of them $\angle KAM$, $\angle KBM$, are equal to two right angles; the other two $\angle AKB$, $\angle AMB$ are equal to two right angles; but the angles $\angle DEG$, $\angle DEF$ are likewise equal to two right angles; therefore the angles, $\angle AKB$, $\angle AMB$, are equal to $\angle DEG$, $\angle DEF$, of which $\angle AKB$ is equal to $\angle DEG$, wherefore the remaining angle $\angle AMB$ is equal to the remaining angle $\angle DEF$: In like manner, the angle $\angle LNM$ may be demonstrated to be equal to the angle $\angle DFE$; and therefore the remaining angle $\angle MLN$ is equal to the remaining angle $\angle EDF$; wherefore the triangle LMN is equiangular to the triangle DEF : and it is described about the circle ABC ; which was to be done.

That part of the demonstration, which proves the three sides of the triangle ABC to meet each other, has been hitherto omitted, and therefore the problem not completely solved.

Prop. 5. Problem. "To describe a circle about a given triangle."

Let the given triangle be ABC , it is required to describe a circle about ABC .

Bisect



Bisect AB , AC , in the points D , E , and from the points D , E , draw the straight lines DF , EF at right angles to AB , AC (prop. 11. B. I.); and join DE ; the two angles EDF , DEF are less than the two angles FDA , FEA , but FDA , FEA are two right angles, therefore FDE , DEF , are less than two right angles; but when a straight line falling upon two other straight lines makes the two interior angles upon the same side, taken together, less than two right angles, these two straight lines being continually produced, shall at length meet on that side (prop. B. B. I.); therefore DF , EF shall meet in some point as F ; join FA ; and if the point F be not in the line BC , join FB , FC : Because AD is equal to DB , and DF common, and at right angles to AB , the base FA is equal to the base FB . In the same manner it may be shewn, that FA is equal to FC ; and therefore FB , which is equal to FA , is likewise equal to FC ; and therefore the circle described about the center F , at the distance of one of them

them shall pass through the extremities of the other two; and be described about the triangle ABC; which was to be done.

This demonstration is like that of Simson, who has considerably improv'd this proposition, in every other respect but that the meeting of the lines DF, EF is demonstrated in a different manner. He endeavours to prove it thus, "DF and EF produced shall meet one another: For, if they do not meet, they are parallel; wherefore AB, AC, which are at right angles to them are parallel; which is absurd;" it is neither very clear, nor indeed strictly true, that if DF, EF, are parallel, BA, AC are parallel. They may be either parallel, or they may be one and the same line; each of which would be indeed absurd; but these are proofs, which do not immediately follow from any demonstration, which has yet occur'd in the Elements, and therefore are not proper in this place.

Cor. "And it is manifest, that when the center of the circle falls within the triangle, each of its angles is less than a right angle, each of them being in a segment greater than a semicircle. But, when the center is in one of the sides of the triangle, the angle opposite to the side beyond which it is, being in a segment less than a semicircle, is greater than a right angle. So that if the given angle be less than a right angle, the straight lines DF, EF, fall within the triangle; if it be a right angle, they fall upon BC; and if it be greater than a right angle, they fall beyond BC. q. e. d."

Instead of criticising and correcting this corollary, the editors of Euclid would have done a greater service to the elements, if they had entirely removed

ved it; as it does not belong to this proposition; but is demonstrated in prop. 31. B. III.

Prop. 7. Problem. "To describe a square about a given circle."

Before it can be inferred that the figure constructed is a square, it ought to be proved that its adjacent sides meet each other, which has not been done in the demonstration of this problem. The same defect occurs in several other demonstrations, which shews the necessity of prop. C. B. I. to prove, that when a straight line meets another straight line, it shall meet every straight line, which is parallel to it.

Prop. 12. Problem. "To describe an equilateral and equiangular pentagon about a given circle."

We observe the same omission in the demonstration of this as of the seventh proposition; which may be supplied by joining the points in which any two adjacent sides of the pentagon touch the circle, and shewing the two interior angles on the same side of this line to be less than two right angles, as was done in prop. 3. of this book.

Cor. to prop. 15. From hence it is manifest, that the side of the hexagon is equal to the semidiameter of the circle. And if we draw through the points A, B, C, D, E, F, tangents to the circle, an equilateral and equiangular hexagon will be described about the circle, as is manifest from what has been said concerning the pentagon. And so likewise may a circle be inscribed and circumscribed about a given hexagon: which was to be done.

We

We find only the former part of this in Campanus's translation; and indeed the whole may be removed, as it would be equally clear without the corollary. The same may be said of the following observations annexed to prop. 16.

“ If, according to what was said of the pentagon,
“ right lines are drawn through the divisions of
“ the circle touching it, there will be described
“ about the circle an equilateral and equiangular
“ quindecagon. And moreover, from what has
“ been said concerning the pentagon, a circle may
“ be described about a given equilateral and equi-
“ angular quindecagon.”

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BOOK V.

DEFINITIONS.

DEFINIT. 4. Λογον εχειν προς αλληλα λεγεται
τα μεγαθη, α δυναται πολλαπλασιαζομενα αλληλων υπε-
ρεχειν.

“Magnitudes are said to have a ratio to each
“other, when either of them may be *multiplied* so
“as to exceed the other.”

The word πολλαπλασιαζομενα, “multiplied,” occurs
only in this and two other places in the books of
Euclid upon plane geometry, and I believe in none
of his other books, excepting those which treat of
numbers. For to *multiply* is an operation of arith-
metic, not of geometry. And therefore, in geo-
metry,

metry, Euclid always uses the expression *εἰληφθῶσι πολλαπλάσια* "let multiples be taken." For these and other reasons which will appear when we examine prop. 8. of this book, we may suspect this definition and the other passages where this word occurs to have been corrupted at least, if not to be altogether spurious.—Def. C. B. I. "A magnitude "is said to be finite, when any other magnitude "may be taken such a number of times as to be "greater than it," supplies the place of this definition, which may therefore be entirely expunged.

The two following are the only definitions in this book, which furnish equations and are referred to as principles of reasoning: the rest are rather descriptions of certain technical expressions, than geometrical definitions.

Definit. 5. "Magnitudes are said to be in the "same ratio, the first to the second and the third "to the fourth, when any equimultiples whatever "of the first and third being taken, and any equi- "multiples whatever of the second and fourth, if "the multiple of the first be greater than that of "the second the multiple of the third is greater "than that of the fourth, if equal, equal, and if "less, less."

Definit. 7. "But when of the equimultiples, "the multiple of the first exceeds the multiple of "the second, but the multiple of the third does not "exceed the multiple of the fourth, the first is said "to have to the second a greater ratio than the "third to the fourth."

P R O P O S I T I O N S.

Cor. to prop. 4. "From this it is manifest, that
 "if four magnitudes be proportionals they shall
 "be proportionals also when taken inversely."

In most editions this corollary is preceded by a demonstration of inversion: but Campanus has no demonstration of it; and Dr. Gregory observes that it is not demonstrated in some manuscripts.

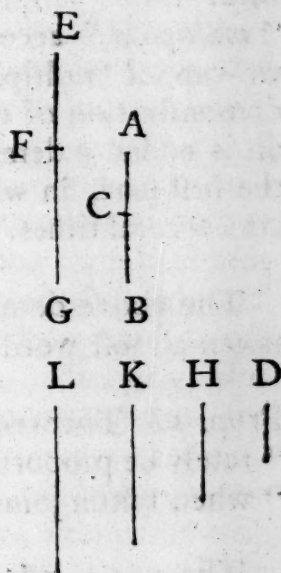
Euclid has said nothing of inversion in B. VII. which teaches the proportions of numbers, and which bears a strong analogy to this book. Would not Euclid have employed inversion in the second case of prop. 7. of this book, viz. "Equal magnitudes have the same ratio to the same magnitude, and the same has the same ratio to equal magnitudes," if inversion had gone before it? The application of inversion to the second case of prop. 9, viz. "Magnitudes, which have the same ratio to the same magnitude, are equal to each other; and magnitudes to which the same has the same ratio are equal to each other," is too obvious for Euclid or any person to overlook. The use of inversion in these cases would induce us to place it among the propositions of this book; but that we should thereby increase the number in order to lessen the difficulty of our theorems. Brevity is particularly useful in the fifth book, where the learner will be impatient to know the use of those general theories, which till he sees their application he never thoroughly understands. Therefore he will do well to pass over all the propositions which are not Euclid's. He will by this means acquire the doctrine of proportion in an easy and rational manner.

manner, and will after readily comprehend or from his own judgment demonstrate many theories not expressed in the elements, but which are easily deducible from the doctrines there laid down.

Prop. 8. Theorem. "Of unequal magnitudes the greater has a greater ratio to the same than the less has; and the same magnitude has a greater ratio to the less than it has to the greater."

Let AB, CB, be unequal magnitudes, of which AB is the greater; and let D be any magnitude whatever: AB has a greater ratio to CB, than to AB.—Let EF, FG, be taken equimultiples of AC, CB, so that the multiple of each of them be greater than D; and let H be taken double of D; K its triple; and so on continually, till the multiple of D be that which first becomes greater than FG; let L be the multiple of D, which first becomes greater than FG, and K the multiple of D which is next less than L.

Therefore because L is the multiple of D, which first becomes greater than FG; K is not greater than FG; that is, FG is not less than K. And because EF is the same multiple of AC that GF is of CB; FG is the same multiple of CB that EG is of AB (prop. I. B. V.); wherefore EG and FG are equimultiples of AB and CB; and it was shewn that FG is not less than K; and, by the construction EF is greater than D; therefore the whole EG is greater than



K and

K and D taken together, that is than L; and FG is not greater than L. But EG and FG are equimultiples of AB, and CB, and L is a multiple of D; therefore AB has to D a greater ratio than CB has to D. (Def. 7. B. V.)

Also D has to CB a greater ratio than it has to AB. For, the same construction being made, it may be shewn in like manner, that L is greater than FG, but that it is not greater than EG: But L is a multiple of D; and FG, EG are equimultiples of CB, AB; therefore D has to CB a greater ratio than it has to AB (def. 7. B. V.) Wherefore, "Of unequal magnitudes, &c." q. e. d

The construction of this proposition both in the Greek copies and in our translations is more minute than there is any occasion for. Indeed we need not be surprized at any error in the Greek construction, the author of which was not even acquainted with the language of Euclid; having several times employed the word *πολλαπλασιαζόμενον* "*multiplied*," according to our translators, whereas we cannot multiply but by a number. After the demonstration of the second part of this proposition is added a demonstration of a different case of the first part, in which the verb *πολλαπλασιαζομαι* occurs several times.

The above demonstration of this proposition is taken almost word for word from Simson.

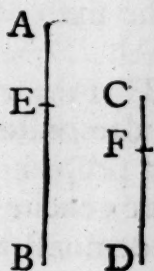
Prop. 18. Theorem. "If magnitudes taken separately be proportionals, they shall be proportionals when taken jointly."

The original demonstration of composition is allowed to be an imperfect one: and as no good demon-

monstration of composition has ever been substituted in its place, perhaps the following, which is similar to Euclid's demonstration of the composition of numbers, will be the best which can be given.

Let AE be to $\frac{1}{2}$ B, as CF to FD;
I say AB shall be to EB, as CD to FD.

Because AE is to EB as CF to FD, by alternation AE is to CF as EB to FD (prop. 16. B. V.), and as one of the antecedents is to its consequent so shall all the antecedents taken together be to all the consequents (prop. 12. B. V.) that is, AE, EB, taken together, shall be to CF, FD, taken together, as EB is to FD: And by alternation AB shall be to EB, as CD to FD. Therefore "If magnitudes taken separately, &c." q. e. d.



/E

Prop. 13. Theorem. "If a whole magnitude be
"to a whole as a magnitude taken from the first to
"a magnitude taken from the second, the remain-
"der shall be to the remainder as the whole to the
"whole."

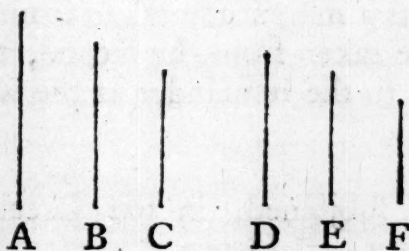
To this is subjoined, in the Greek, the following corollary; "Hence it is manifest, that if
"four magnitudes be proportionals, they shall also
"be proportionals by conversion." Clavius, and after him Simson, have demonstrated this corollary upon different principles; not observing that it follows by alternation from the proposition; as is shewn in a demonstration preceding the corollary in the original: I shall say nothing of the inaccuracies in that demonstration, because it will be better entirely to omit both it and the corollary following it.

Prop.

Prop. 20. Theorem. "If there be three magnitudes, and other three which taken two and two have the same ratio; if the first be greater than the third, the fourth shall be greater than the sixth; if equal, equal; and if less, less."

This theorem is of no other use than to assist in the demonstration of prop. 22. of this book. But that proposition may be demonstrated in a much more concise and easy manner without it; by which six demonstrations may be comprized in one.

Prop. 22. Theorem. "If there be any number of magnitudes and as many others, which taken two and two in order have the same ratio, the first shall have to the last of the first magnitudes the same ratio which the first of the others has to the last."



Let A, B, C be any three magnitudes, and D, E, F an equal number, and let A be to B as D to E and B to C as E to F, I say A shall be to C as D to F.

Because A is to B as D is to E, by alternation A shall be to D, as B to E, (prop. 16. B. V.); for the same reason because B is to C as E to F, B shall be to E as C to F: but ratios that are the same

same to the same ratio are the same to one another (prop. 11. B. V.); therefore A shall be to D as C to F; and by alternation, A shall be to C, as D to F (prop. 16. B. V.) And if the number of magnitudes be greater than three, we may proceed in the same manner with the first, third and fourth magnitudes, of each series, as with the first, second and third; and so on to any number of magnitudes whatever. Therefore "If there be any number of magnitudes, &c." q. e. d.

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BOOK VI.

DEFINITIONS.

DEFINIT. I. "Similar rectilineal figures
"are those which have their several angles
"equal, each to each, and the sides about the
"equal angles proportionals."

It does not as yet appear that rectilineal figures may have their angles equal each to each, and also the sides about the equal angles proportionals. This definition therefore is not consistent with that excellent rule "that a definition should be better known than the thing defined."

In giving a definition of similar triangles, the abovementioned inaccuracy may be very easily avoided:

avoided: the equality of the angles being alone sufficient to constitute similar triangles, as is evident from prop. 4. of this book. Therefore let their definition be;

Similar triangles are those which have their angles equal each to each.

But the similitude of rectilineal figures of more than three sides does not depend upon so simple a test. Their angles may be equal each to each, and yet the sides about their equal angles not proportionals. We may however by means of similar triangles reduce their definition to the same principle. The triangle is as it were the proper element of other rectilineal figures, it is their common measure; as unit is of all integer numbers: by the triangle we are enabled to describe a parallelogram equal to any rectilineal figure in the first book; and in the fifth rectilineal figures are constructed similar to other rectilineal figures, and all their properties are investigated by dividing them into triangles. It will not therefore be improper to employ the same figure in their definition, as follows;

Definit. A. Similar rectilineal figures of more than three sides are those which may be divided into an equal number of similar triangles, similarly situated.

Some alterations will take place in prop 18. and 20. of this book, in consequence of this definition; of which hereafter.

Definit. 5. Λογὸς ἐκ λόγων συγκείμεναι λέγεται ὅταν αἱ τῶν λόγων πηλικότητες ἐφ' αὐτὰς πολλαπλασιασθεῖσαι ποίωσι τινὰ.

An account of the different interpretations which different writers have put upon this definition may

be found in Simson's note upon prop. 23. of this book, where likewise are offered many arguments which invalidate its authenticity. This is the third passage where we meet with the verb *πρόσθησθαι*.

PROPOSITIONS.

Cor. to prop. 8. "From this it is manifest, that
 "if in a right angled triangle a perpendicular be
 "drawn from the right angle to the base, it is a
 "mean proportional between the segments of the
 "base, and moreover the side adjacent to either of
 "the segments of the base is a mean proportional
 "between the whole base and that segment."

In the Greek demonstration of the proposition, after proving the triangles to be equiangular, it is expressly shewn that the perpendicular is a mean proportional between the segments of the base. Though the original definition of similar figures requires, that both their angles should be equal, each to each, and the sides about the equal angles proportionals; yet, after prop. 4. which proves that equiangular triangles have their sides about the equal angles proportionals, this minuteness is not necessary. And it is not improbable that the author had particularly in view prop. 13. of this book, which depends upon this property of the right angled triangle, hereby precluding the occasion of a corollary.

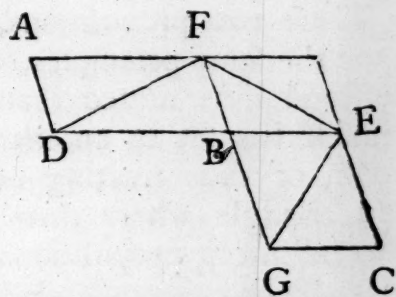
But, I believe, both that part of the demonstration and the corollary likewise may be omitted without any inconvenience. If, in any subsequent proposition, particular inferences are made from the properties of a right angled triangle contained
 in

in this, is not the demonstration of that proposition the proper place for deducing them? With little alteration prop. 13. may be demonstrated, in a very easy manner, without these preliminary steps.

Prop. 14. and 15. Theorems. Equal parallelograms or equal triangles, which have one angle of the one equal to one angle of the other, have their sides about the equal angles reciprocally proportional: And parallelograms or triangles, which have one angle of the one equal to one angle of the other, and their sides about the equal angles reciprocally proportional, are equal to one another.

Let AB, BC, be equal parallelograms, and DFB, BEG, equal triangles which have the angles at B equal, and let the sides DB, BE, be placed in the same straight line; FB, BG shall also be in one straight line (prop. 14. B. I.) The sides about the equal angles are reciprocally proportional; that is, DB is to BE, as GB to BF.

Complete the parallelogram FE, and join FE. And because the parallelogram AB is equal to BC, and FE is another parallelogram, AB is to FE, as BC



to FE (prop. 7. B. V.). But as AB to FE, so is the base DB to BE, and as BC to FE, so is the base GB to BF (prop. 1. B. VI.); therefore as DB to BE, so is GB to BF. (prop. 11. B. V.)

For the same reasons the triangle DBF is to the triangle FBE, as GBE is to FBE, and likewise their bases DB to BE, as GB to BF. Wherefore the
sides

sides of the parallelograms AB, BC, and of the triangles DBF, GBE, about their equal angles are reciprocally proportional.

But let the sides about the equal angles be reciprocally proportional, viz. as DB to BE, so GB to BF; the parallelogram AB is equal to the parallelogram BC, and the triangle DBF to the triangle GBE.

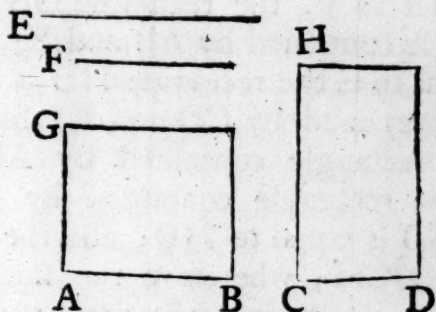
Because DB is to BE, as GB to BF; and as DB to BE so is the parallelogram AB to the parallelogram FE; and as GB to BF, so is the parallelogram BC to the parallelogram FE (prop. 1. B. VI.), AB shall be to FE, as BC to FE (prop. 11. B. V.); wherefore the parallelogram AB is equal to the parallelogram BC (prop. 9. B. V.); and so likewise is the triangle DBF equal to the triangle GBE, which are halves of the equal parallelograms AB, BC. Therefore equal parallelograms, or equal triangles, &c. q. e. d.

These two propositions, which depend entirely upon the same principles, will be much sooner understood thus united than when separate. Euclid himself has set us an example in prop. 1. of this book, of thus treating of the parallelogram and and triangle where their properties agree. The two following propositions which depend upon this may also be comprehended under the same demonstration.

Prop. 16. and 17. Theorems. If three, or four straight lines be proportionals, the rectangle contained by the extremes is equal to the rectangle contained by the means, or the square of the mean; And, if the rectangle contained by the extremes be

be equal to the rectangle contained by the means, or the square of the mean, the three, or four straight lines are proportionals.

Let AB , CD , E , F , be four straight lines; and let AB be to CD , as E to F ; the rectangle contained by AB and F is equal to the rectangle contained by CD and E .



At the point A in the straight line AB , draw AG at right angles to AB , and make AG equal to F and complete the rectangle GB : Also at the point C in the straight line CD draw CH at right angles to CD ; make CH equal to E , and complete the rectangle HD . Then because AB is to CD , as E to F ; AB is to CD , as CH to AG (prop. 7. B. V.); therefore the sides about two equal angles of the parallelograms GB , HD , are reciprocally proportional; and therefore the rectangle GB is equal to the rectangle HD (prop. 14. B. VI.): but the rectangle GB is equal to the rectangle contained by AB and F because AG is equal to F : also because CH is equal to E , the rectangle HD is equal to the rectangle contained by CD and E . In the same manner if AB be to CD , as CD to F , it may be proved that the rectangle contained by AB and F is equal to the rectangle contained by CD and CD that is to the square of CD . Therefore

fore the rectangle contained by the extremes is equal to the rectangle contained by the means or to the square of the mean.

And if the rectangle contained by the extremes AB and F be equal to the rectangle contained by CD and E, AB shall be to CD, as E to F.

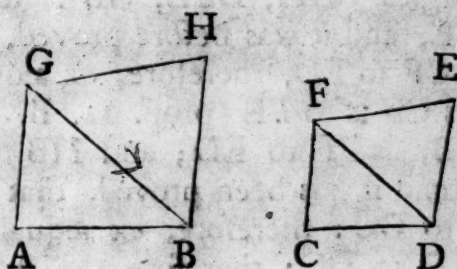
For the same construction being made, because AG is equal to F, the rectangle GB is equal to the rectangle contained by AB and F; also because CH is equal to E the rectangle HD is equal to the rectangle contained by CD and E: but by supposition, the rectangle contained by AB and F is equal to the rectangle contained by CD and E, therefore GB is equal to HD; and the angle BAG is equal to DCH; wherefore the sides about the equal angles are reciprocally proportional, that is, AB is to CD as CH to AG (prop. 14. B. VI.); but CH and AG are equal to E and F, therefore AB is to CD as E to F (prop. 7. B. V.).

And if CD be equal to E, the rectangle HD is equal to the square of CD, and therefore in that case AB is to CD, as CD to F. Therefore if three, or four straight lines, &c. q. e. d.

Prop. A. Theorem. Similar polygons have their angles equal, each to each; and the sides about the equal angles proportionals.

Let AGHB, CFED, be similar polygons, and let them be divided into the similar, and similarly situated triangles, AGB, CFD; GHB, FED (Def. A. B. VI.); the angles BAG, AGH, GHB, HBA, in the polygon AGHB are equal to the angle DCF, CFE, FED, EDC, in the polygon CFED.

The



Because the triangles BAG, DCF, are similar, and similarly situated, the angle BAG is equal to DCF, and the angle AGB to CFD, and also GBA to FDC. Also because the triangle GHB, is similar and similarly situated to the triangle FED, it has the angles BGH, GHB, HBG, equal to the angles DFE, FED, EDF, each to each. And because the angle AGB is equal to CFD, and the angle BGH to DFE; the whole angle AGH is equal to CFE: also because the angle HBG, is equal to EDF, and the angle GBA to FDC, the whole angle HBA is equal to EDC; and it was before proved that the angles BAG, GHB, are equal to the angles DCF, FED, each to each; therefore the polygon BAGH is equiangular to the polygon DCFE.

And the sides about the equal angles in the polygon BAGH are proportional to the sides about the equal angles in the polygon DCFE; that is, BA is to AG, as DC to CF, and AG to GH, as CF to FE, and GH to HB, as FE to ED, and HB to BA, as ED to DC.

For in the similar and similarly situated triangles BAG, DCF, the sides about the equal angles are proportionals, that is, BA is to AG, as DC to CF; and AG is to GB, as CF to FD, and GB is to BA, as FD to DC: likewise in the similar and similarly

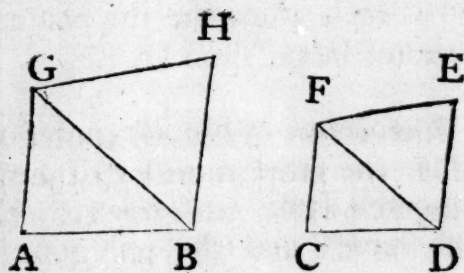
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situated

situated triangles BGH, DFE, GB is to GH, as DF is to FE, and it was before proved, that AG is to GB as CF to FD, therefore, ex æquali, AG is to GH, as CF is to FE (prop. 22. B. V.); also GH is to HB, as FE to ED; and HB to BG, as ED to DF, and it has been proved, that GB is to BA, as FD to DC: therefore, ex æquali, HB is to BA, as ED to DC: therefore the sides about the equal angles in the polygon BAGH, are proportional to the sides about the equal angles in the polygon DCFE. Therefore, Similar polygons, &c. q. e. d.

Prop. 18. Problem. "Upon a given straight line to describe a rectilineal figure, similar and similarly situated to a given rectilineal figure."

Let AB be a given straight line, and DEFC a given rectilineal figure, it is required to describe upon the given straight line AB a rectilineal figure similar and similarly situated to the rectilineal figure DEFC.



Join FD: and at the points A and B in the straight line AB make the angles BAG, ABG, equal to the angles DCF, CDF; each to each, therefore the remaining angle AGB is equal to the remaining angle CFD, and the triangle AGB is similar to the triangle DGE. Again, at the points G and B in the straight line GB make the angles, BGH,

BGH, GBH equal to the angles DFE, FDE, each to each, therefore the remaining angle GHB is equal to the remaining angle FED, and the triangle GHB is similar to the triangle FED (prop. 4. B. VI.): therefore the two triangles BAG, GHB are similar, and similarly situated to the triangles DCF, FED, therefore the rectilineal figure AGHB is similar and similarly situated to the rectilineal figure CFED, and it is described upon the given straight line AB: which was required to be done.

Cor. to prop. 19. "From this it is manifest that if three straight lines are proportionals, as the first is to the third, so is a triangle described upon the first to a similar and similarly described triangle upon the second; for it was shewn that as AB (the first) is to BG (the third), so is the triangle upon AB, (the first), to the similar triangle upon BE (the second). q. e. d."

This corollary only expresses, in different words, the sense of the proposition, viz. "that similar triangles have to each other the duplicate ratio of their homologous sides."

Prop. 20. Theorem. "Similar polygons may be divided into the same number of similar triangles, having the same ratio to each other that the polygons have; and the polygons have to one another the duplicate ratio of that which their homologous sides have."

That part of this proposition, which relates to dividing similar polygons into the same number of similar triangles may be omitted, on account of the definition of similar polygons abovementioned.

Cor. 1. to prop. 20. "In like manner it may also be proved that four sided figures are to each other in the duplicate ratio of their homologous sides: *and it has been already demonstrated of triangles*: so that universally similar rectilineal figures are to each other in the duplicate ratio of their homologous sides. And if to A and B we take a third proportional C, A has to C the duplicate ratio that A has to B, but the polygon has to the polygon, and the four-sided figure to the four-sided figure the duplicate ratio that the homologous side has to the homologous side, that is, that A has to B. *And this has been already demonstrated of triangles.*"

Every reader sees, that this tedious corollary is contained in the proposition; and that the word polygon is there applied to all rectilineal figures of more than three sides.

Cor. 2. to prop. 20. Ωστε και καθολικ φανερων, οτι εαν τρεις ειδειαι αναλογον ωσιν, εσαι ως η πρωτη προς την τριτην, κτως το απο της πρωτης ειδθ προς το απο της δευτερας το ομοιον, και ομοιως αναγραφομενον. οπερ εδει δειξαι.

"So that it is also manifest universally, that if
 "three straight lines be proportionals, as the first
 "is to the third, so shall be any *form* described upon
 "the first to one similar and similarly described upon
 "the second: Which was to be demonstrated."

The word ειδθ, "*species*", according to the Latin translators, is first used in this corollary. From definition 14. B. I. we find that *χημα* "*figure*" is the term which Euclid adopted to express inclosed space. An argument, founded upon internal evidence of this nature, to prove the additions and corruptions, made in any work, carries with it great force on all occasions; but it has superior

perior weight in a system, the original author of which limits himself by definitions to particular terms: And of whom it is said, “neque mos fit “Euclidi et mathematicis, a receptâ semel formâ “dicendi post recedere, aut in copiâ verborum inniter ludere”: Savile, Lect. 13. This expression occurs in an unnecessary quotation of this corollary in prop. 25. of this book, and in a few subsequent propositions; of which in their proper place. Besides, the inference made in this corollary does not follow from the proposition; which treats only of *rectilineal* figures or polygons; and does not extend to *forms*, or, more properly speaking, figures in general, as is asserted in the corollary.

Prop. 27. Theorem. “Of all parallelograms “applied to the same straight line, and deficient “by parallelogram *forms* similar, and similarly situated to that which is described upon half of “the line, that which is applied to half of the line, “and is similar to its defect, is the greatest.”

Prop. 28. Problem. “To a given straight line “to apply a parallelogram equal to a given rectilineal figure, deficient by a parallelogram *form*, similar to that which is given. But the given rectilineal figure to which the parallelogram to be applied is to be equal, must not be greater than “that applied to half of the line, having its defect “upon the other half similar to the given parallelogram.”

Prop. 26. Problem. To a given straight line to apply a parallelogram equal to a given rectilineal figure, exceeding by a parallelogram *form*, similar to one given.

Tacquet

Tacquet and Dechales left the two last-mentioned propositions out of their editions of the elements: Simson contends for retaining them on account of their utility; but whether they are entitled to a place in the elements upon that ground, I shall not pretend to determine. Euclid does not seem to have thought so; as we may almost conclude with certainty from the language, that they are not his. Not only in the enunciation of them, but throughout their demonstration, the word *εἶδος* is almost every where employed instead of the geometrical term *ῥημα*.

Prop. 30. Problem. Τὴν τινθεσαν εὐθειαν πεπερασμένην, ἀκρον καὶ μέσον λόγον τέμνειν.

“To cut a terminated straight line *proposed* in “extreme and mean ratio.”

I find the word *τινθεσαν* instead of *δοθεισαν* in all the Greek copies; which is interpreted as above, by Campanus and Tartalea.

In the first demonstration of this proposition, which depends upon prop. 20, the word *εἶδος* occurs, and in the conclusion of the demonstration, *ὅπερ εἶδει δεῖξαι*, “which was to be demonstrated,” is employed instead of, *ὅπερ εἶδει ποιῆσαι*, “which was “to be done”; with which Euclid concludes all his problems. The second demonstration of it which refers to prop. 11. B. II. is very accurate; and indeed this problem is solved so easily from that proposition, that any reader would make it out without a demonstration.

Prop. 31. Theorem. “In right angled triangles, “the *form* upon the side, subtending the right angle,
“gle,

“gle, is equal to the similar and similarly describ’d
“forms upon the sides containing the right angle.”

Throughout this proposition the word *υδδς* is used instead of *ζημα υδδγραμμων*. But the proposition is superfluous, for it is evident that all similar rectilinear figures, described upon any straight lines, are to each other in the same proportion as any other similar and similarly described rectilinear figures, upon the same straight lines: but the square of the hypotenuse is equal to the squares of the sides containing a right angle; therefore any other rectilinear figure, described upon the hypotenuse, is equal to the similar and similarly described figures upon the sides containing the right angle.

Prop. 27.—31. inclusively, are not only inaccurate in themselves, as I have observed, but they also interrupt the chain of reasoning. There is an obvious connection between prop. 26. and prop. 32. as has been observed by Simson and other geometers. Therefore if these propositions should remain in the elements, they should be placed like other unconnected propositions at the end of the book.

THE END.

DECLARATION

I, the undersigned, do hereby declare that the foregoing is a true and correct copy of the original as the same appears in the records of the Court.

Given under my hand and the seal of the Court at the City of New York, this 1st day of January, 1901.

CLERK OF THE COURT

By _____

Deputy Clerk of the Court

12

Attest: _____

Notary Public for the State of New York

My commission expires _____

Attest: _____

Notary Public for the State of New York

My commission expires _____

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